Introduction to Communicating Sequential Process (CSP) (Lecture 2)

Mannheim, September 2007

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- Basic constructs laws
- Choices
  - Internal (nondeterministic) choice
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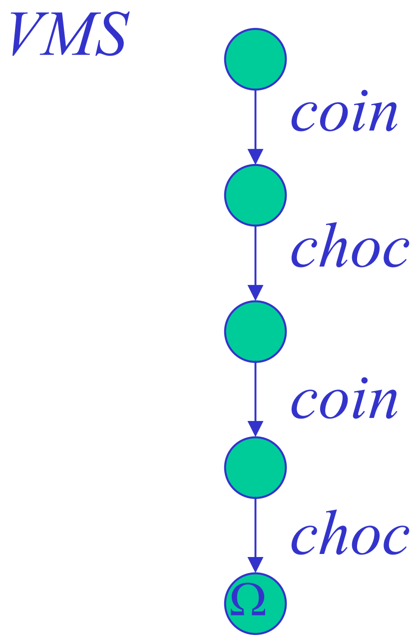
Pictures

- It may be useful to make a pictorial representation of a process as a tree structure, where
  - nodes are states
  - arrows are transitions between states
- The tree includes a starting state

Pictures

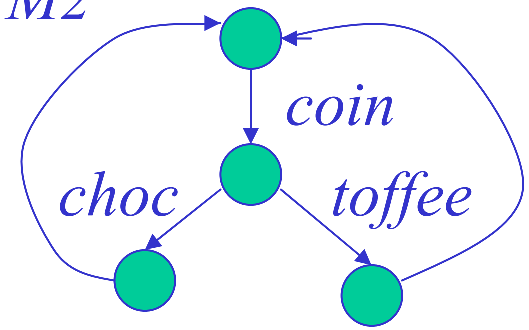
$$VMS = (coin \rightarrow choc \rightarrow coin \rightarrow choc \rightarrow STOP)$$

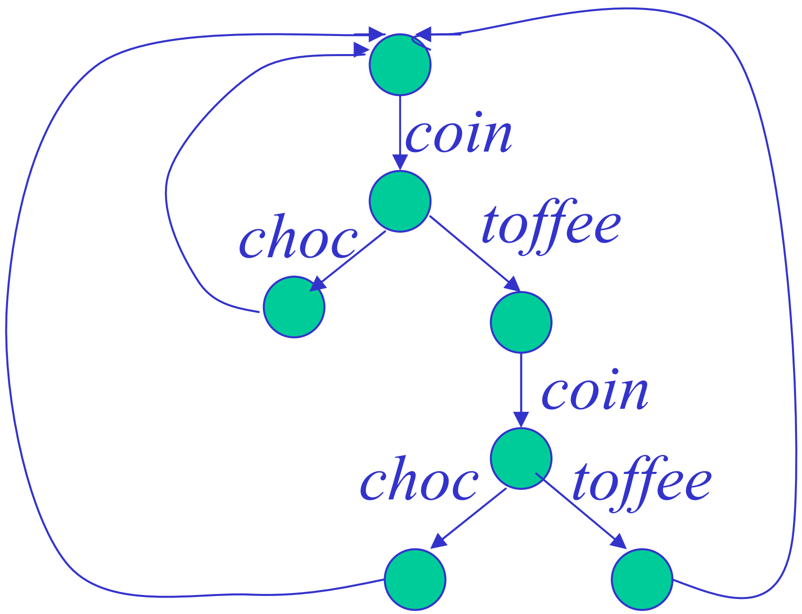


Pictures

$$VM2 = \mu X : \{coin, choc, toffee\} \bullet$$
$$coin \rightarrow (choc \rightarrow X \mid toffee \rightarrow X)$$

VM2





Same process and different pictures => proofs of equality are difficult to conduct

Pictures

- Another problem with pictures: processes with a very large of infinite number of states.
- Example : Rocket

$\alpha Rocket = \{up, down, around\}$

$Rocket = R_0 = (up \text{ -> } R_1 \mid around \text{ -> } Rocket)$

$R_{n+1} = (up \text{ -> } R_{n+2} \mid down \text{ -> } R_n)$

Pictures: Exercise

1. Express Rocket as a tree sturcture.

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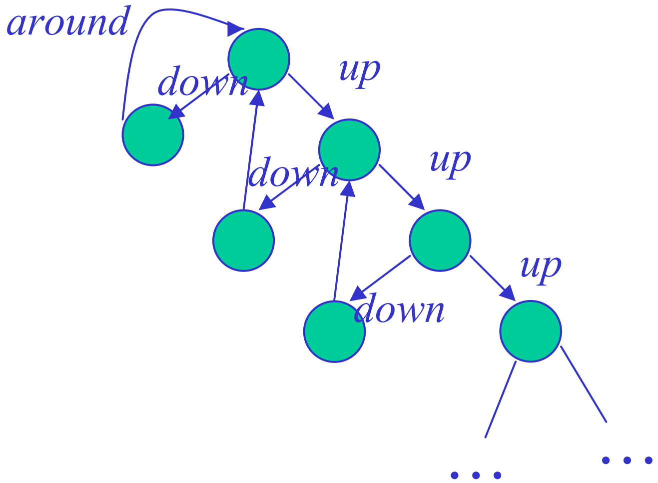


Algebraic Laws of CSP

- There are many different ways of describing the same behaviour

  $(x \text{ -> } P \mid y \text{ -> } Q) = (y \text{ -> } Q \mid x\text{-> } P)$

- We must learn to recognize which expressions describe the same object and which not, as in arithmetic.

  $(x + y) = (y + x)$

- CSP has a set of algebraic laws that defines its semantics and can be used to prove program equivalence.

Some Laws of CSP

- L1: If $A = B$ then $(x{:}A \to P(x)) = (y{:}B \to P(y))$
- Consequences:
  - $STOP \neq (d \to P)$

    Proof:

$$
\begin{aligned}
STOP &= (x{:}\{\} \to P) && \textit{by definition} \\
&\neq (x{:}\{d\} \to P) && \textit{because } \{\} \neq \{d\} \\
&= (d \to P) && \textit{by definition}
\end{aligned}
$$

Some Laws of CSP

- L1: If $A = B$ then $(x{:}A \to P(x)) = (y{:}B \to P(y))$
- Consequences:
  - $(c \to P) \neq (d \to Q)$ *if* $c \neq d$

    Proof ?

Some Laws of CSP

- L1: If $A = B$ then $(x{:}A \rightarrow P(x)) = (y{:}B \rightarrow P(y))$
- Consequences:
  - $(c \rightarrow P \mid d \rightarrow Q) = (d \rightarrow Q \mid c \rightarrow P)$

    Proof:

    Define $R(x) = P$ *if* $x = c$

    $= Q$ *if* $x = d$

    $LHS = (x{:}\{c,d\} \rightarrow R(x))$ *by definition*

    $= (x{:}\{d,c\} \rightarrow R(x))$ *because* $\{c,d\} = \{d,c\}$

    $= RHS$

Some Laws of CSP: Exercise

Prove that

$$\mu X : \{coin, choc, toffee\} \bullet coin \rightarrow (choc \rightarrow X \mid toffee \rightarrow X)$$

$$= \mu X : \{coin, choc, toffee\} \bullet coin \rightarrow (toffee \rightarrow X \mid choc \rightarrow X)$$

Some Laws of CSP: Exercise

Prove that

$$\mu X : \{coin, choc, toffee\} \bullet coin \rightarrow (choc \rightarrow X \mid toffee \rightarrow X)$$

$$= \mu X : \{coin, choc, toffee\} \bullet coin \rightarrow (toffee \rightarrow X \mid choc \rightarrow X)$$

Proof:

Immediate consequence of the previous result.

Internal (nondeterministic) choice

- If P and Q are processes then

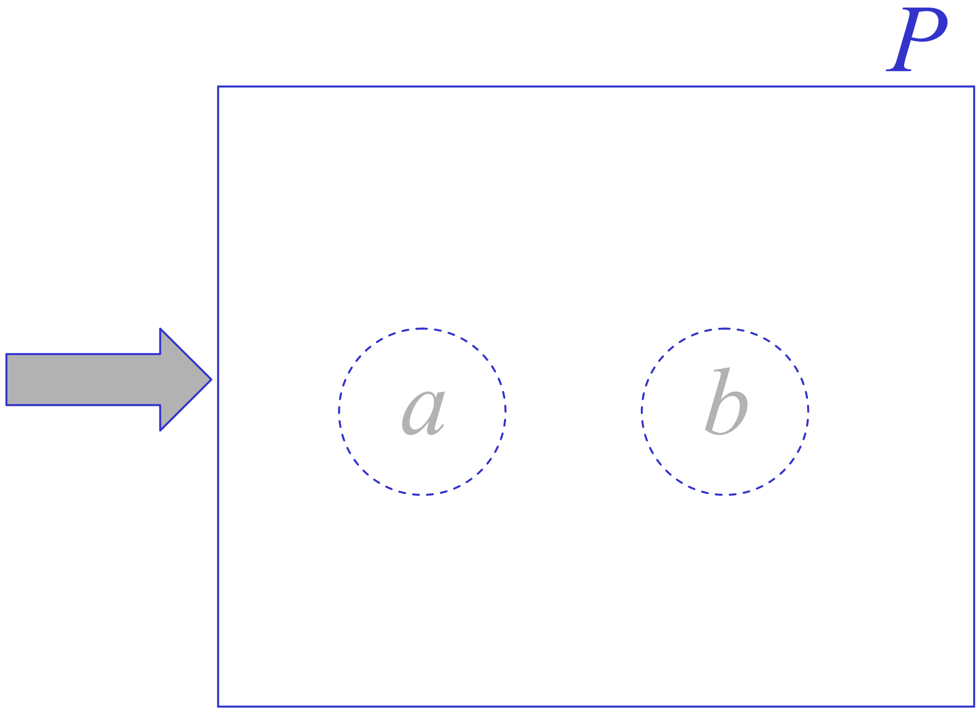
$$P \sqcap Q$$

  denotes the process which is either P or Q; the choice is arbitrary, without the influence of the environment.
- The alphabet of P and Q is the same.
- Internal choice is not a combinator an implementor would wish to use in combining implementations. We are forced to consider it since nondeterminism arises from abstraction.

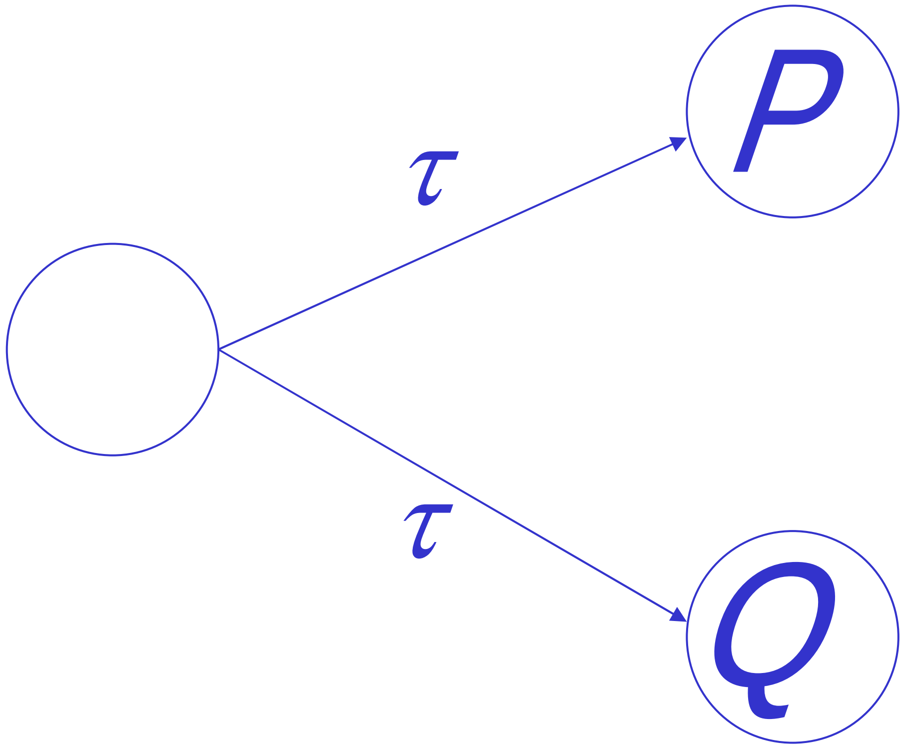
Internal Choice

Notation: $\sqcap$ or |~|

$P = a \text{ -> } Q$
$\quad$ |~| $b \text{ -> } R$



Internal Choice (P |~| Q)



Internal Choice

- How internal choice arises?

  Recall that in sequential programming, abstracting boolean variable x in program

  $$\mathit{if}\ x = \mathit{true}\ \mathit{then}\ P\ \mathit{else}\ Q$$

  gives program

  $$|[\ \mathit{var}\ x \bullet \mathit{if}\ x = \mathit{true}\ \mathit{then}\ P\ \mathit{else}\ Q\ ]|$$

  which is equivalent to a nondeterministic choice between P and Q:

  $$P \sqcap Q.$$

Internal Choice

- Examples:

  Recall the change-giving machine

  $\alpha Chh = \{in5,\ out1,\ out2\}$

  $Chh = (\ in5\ \ \text{->}\ (out2\ \text{->}\ out1\ \text{->}\ out1\ \text{->}\ out1\ \text{->}\ Chh$

  $\quad\quad\quad\quad\quad\quad\quad\quad\quad |\ out1\text{->}\ out2\ \text{->}\ out2\ \text{->}\ Chh)$

  For comparison consider

  $Ch1\ =\ in5\ \text{->}\ Ones$

  $Ones\ =\ out2\ \text{->}\ out1\ \text{->}\ out1\ \text{->}\ out1\ \text{->}\ Ch1$

  $Ch2\ =\ in5\ \text{->}\ Twos$

  $Twos\ =\ out1\text{->}\ out2\ \text{->}\ out2\ \text{->}\ Ch2$

  $Chh\ = Ch1 \sqcap Ch2$

Are they equivalent?

Internal Choice

- Consequences of internal choice
  - 1. The appearance of internal choice in a design means that in reasoning about the design the designer must encompass all possible implementations; bad.
  - 2. The appearance of internal choice in a design allows the implementor a choice in finding an implementation; good. Process $P \sqcap Q$ may be implemented by either P or Q (a choice for the implementor).

Internal Choice: Laws

- Internal choice is set-like (idempotent, commutative and associative)

$$P \sqcap P = P$$

$$P \sqcap Q = Q \sqcap P$$

$$P \sqcap (Q \sqcap R) = (P \sqcap Q) \sqcap R.$$

- A choice that first does a and then makes a choice is indistinguishable from one which first makes the choice and then does a.

$$a \rightarrow (P \sqcap Q) = (a \rightarrow P) \sqcap (a \rightarrow Q)$$

- General choice distributes through internal choice.

$$?x : A \rightarrow (P(x) \sqcap Q(x)) = (?x : A \rightarrow P(x)) \sqcap (?x : A \rightarrow Q(x))$$

Internal Choice: Laws

- Attention: Recursion does not distribute through internal choice. Let $a \neq b$ and

  $$P = \mu X.\ ((a \rightarrow X) \sqcap (b \rightarrow X))$$

  $$Q = (\mu X.\ (a \rightarrow X)) \sqcap (\mu X.\ (b \rightarrow X))$$

  then $P \neq Q$.

  Why?

Internal Choice: Exercise

- A bus company guarantees to provide buses between A and B, but does not guarantee any particular route. There are two routes, 100 and 200, and the passenger is happy to accept either. Model the service offered by the bus company.

Internal Choice: Exercise

- A bus company guarantees to provide buses between *A* and *B*, but does not guarantee any particular route. There are two routes, 100 and 200, and the passenger is happy to accept either. Model the service offered by the bus company.

$$BUS = BUS\_100 \sqcap BUS\_200$$

External choice

- Process $(a \rightarrow A \mid b \rightarrow B)$ is also written

  $$(a \rightarrow A)[](b \rightarrow B)$$

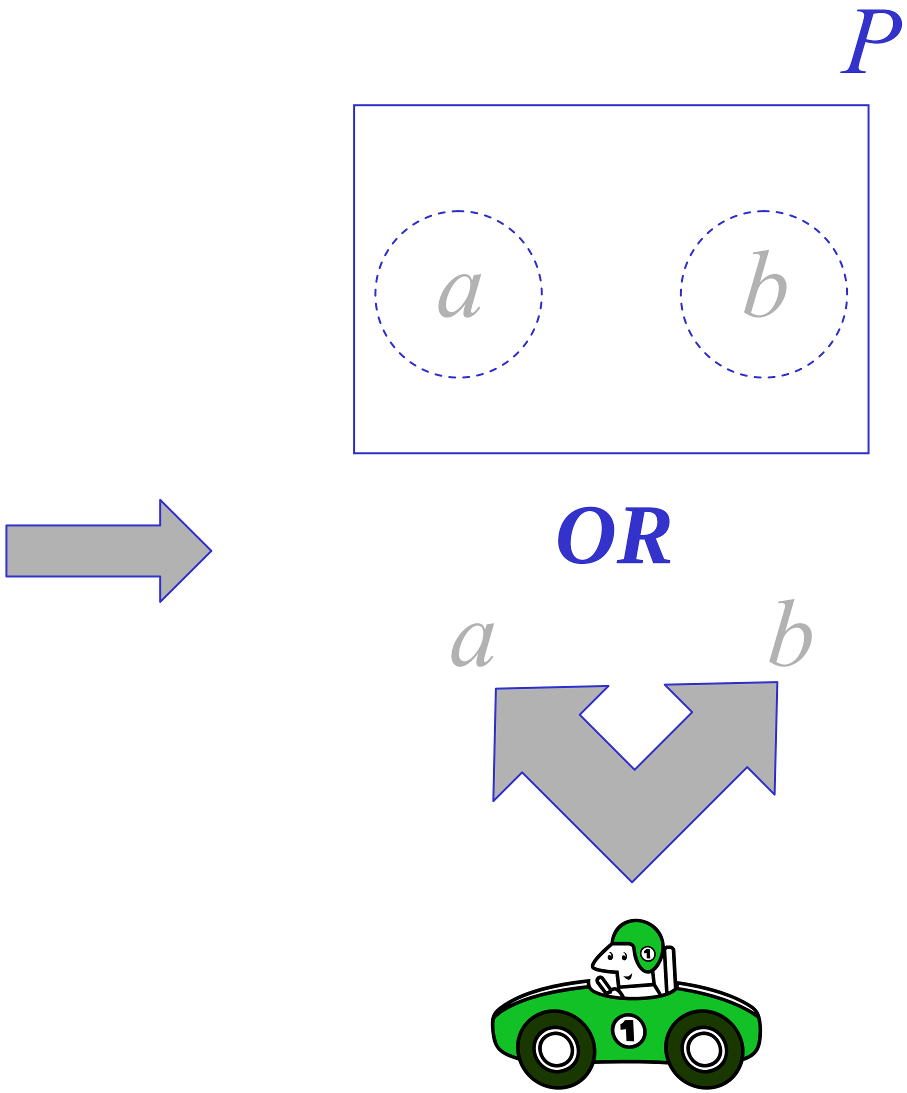
  in which now a and b need not be disinct.
- If P and Q are processes then

  $$P[]Q$$

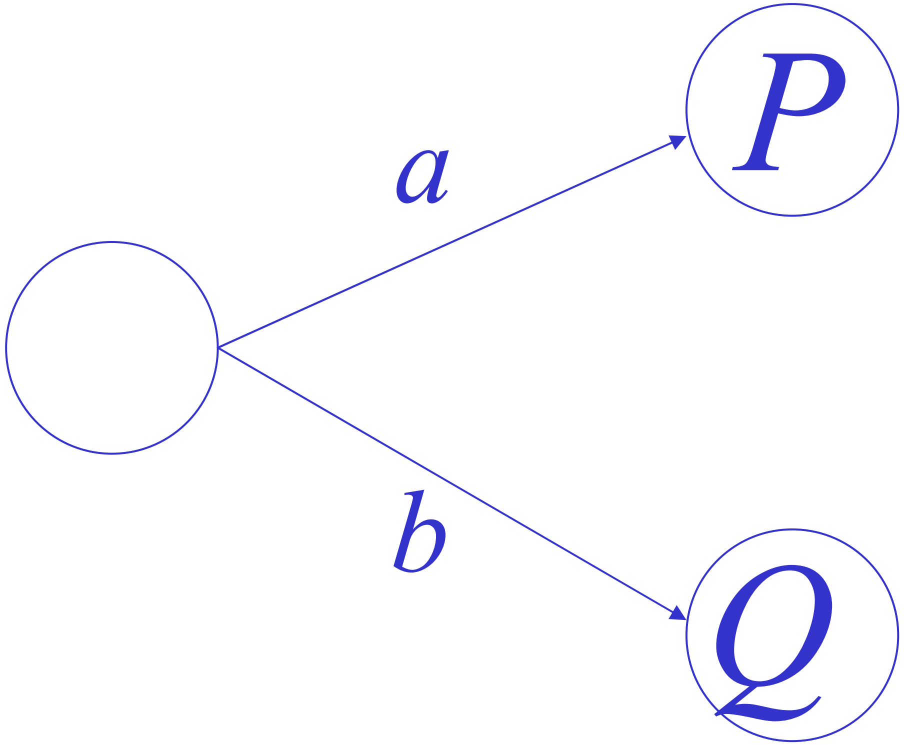
  denotes their external choice whose initial event is determined by the environment: if it is an event of just P then $P[]Q$ behaves like P; if it is an event of just Q then it behaves like Q; if it is an event common to both P and Q then its behaviour is the nondeterministic choice between P and Q.

External Choice

P = *a* -> *Q*
[ ] *b* -> *R*



External Choice (P [ ] Q)



Provided that a is a starting event of P and b a starting event of Q

External choice

- The behaviour of $P[]Q$ is thus:
  - if the environment's first action is not possible for either P or Q then it does not occur;
  - if the environment's first action is possible only for P then P is selected;
  - if the environment's first action is possible only for Q then Q is selected;
  - if the environment's first action is possible for both P and Q then $P \sqcap Q$ is selected.
- Thus,

$$
\begin{aligned}
(a \rightarrow P)\ []\ (b \rightarrow Q) &= (a \rightarrow P \mid b \rightarrow Q),\ a \neq b, \\
&= (a \rightarrow P) \sqcap (a \rightarrow Q),\ a = b.
\end{aligned}
$$

External choice

- Example:
  - A vending machine can offer chocolate and toffee. The user will select the sweet he want and after pay for the appropriate amount.

External choice

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  - A vending machine can offer chocolate and toffee. The user will select the sweet he want, after paying for the appropriate amount.

$$VM = \mu X : \{coin10, coin50, choc, toffee\} \bullet$$
$$(coin10 \rightarrow toffee \rightarrow X \;[]\; coin50 \rightarrow choc \rightarrow X)$$

Comparing internal and external choices

- Observe that the process

$$(a \rightarrow A) \ [] \ (b \rightarrow B)$$

offer its environment a choice between *a* and *b* initially. However the process

$$(a \rightarrow A) \sqcap (b \rightarrow B)$$

offers either *a* or *b* and internally decides which, without the influence of the environment.

- The latter may lead to deadlock if the environment offers only a whilst the former cannot.

External Choice: Laws

- External is set-like (idempotent, commutative and associative)

  $P \,[]\, P = P$

  $P \,[]\, Q = Q \,[]\, P$

  $P \,[]\, (Q \,[]\, R) = (P \,[]\, Q) \,[]\, R.$

.

- STOP is a unit: $P \,[]\, STOP = P$.
- General law:

$$(?x{:}A \to P(x)) \,[]\, (?y{:}B \to Q(y))$$

$$=$$

$$?z{:}\ A \cup B \to \begin{cases} P(z), & z \in A \setminus B \\ Q(z), & z \in B \setminus A \\ P(z) \sqcap Q(z), & z \in A \cap B \end{cases}$$

External choice: Laws

- External choice distributes through internal choice and vice-versa.

  $P \;[]\; (Q \sqcap R) = (P \;[]\; Q) \sqcap (P \;[]\; R)$

  $P \sqcap (Q \;[] R) = (P \sqcap Q) \;[]\; (P \sqcap R)$

- Nevertherless,

  $a \text{ -> } (P \;[]\; Q) \neq (a\text{->}P) \;[]\; (a\text{->}Q)$

  Why?

Input and External Choice

Let *c* be a channel of type $\{e_0, \ldots, e_N\}$

$$c?x \rightarrow P \quad \Longrightarrow \quad \begin{array}{l} c.e_0 \text{ -> } P[e_0/x] \\ [\,] \\ \ldots \\ [\,] \\ c.e_N \text{ -> } P[e_N/x] \end{array}$$

External Choice: Exercise

1. Present a CSP process which captures the first menu of an ATM machine.

External Choice: Exercise

1. Present a CSP process which captures the first menu of an ATM machine.

Menu = *withdraw -> P*
[] *balance -> Q*
[] *deposit -> R* …

Conditional Choice

- If b is a predicate and P and Q are processes then

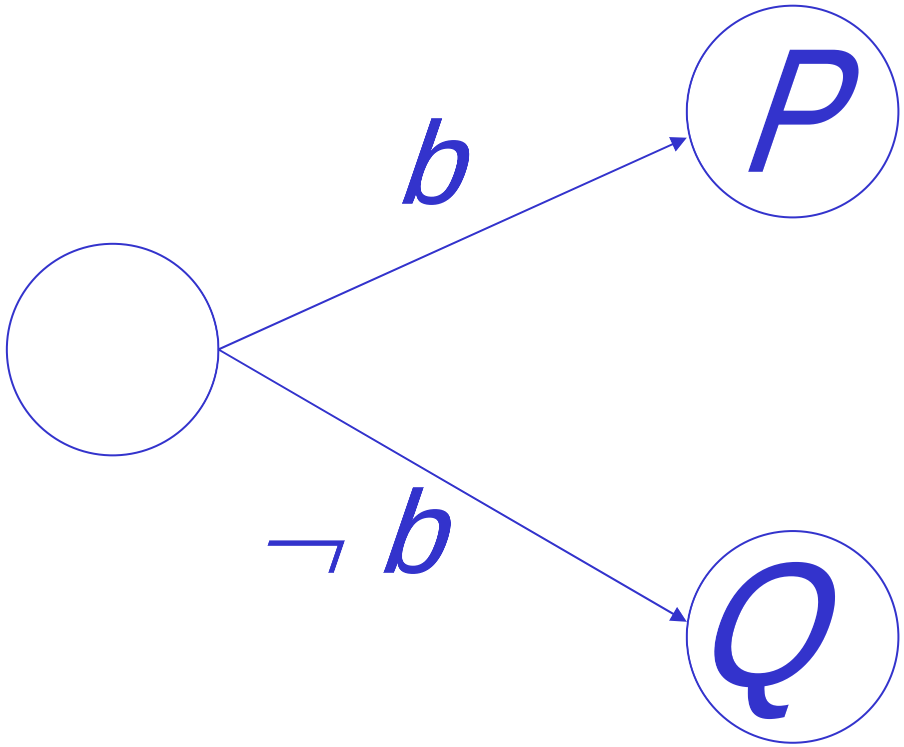
$$P \lhd b \rhd Q$$

  denotes the process which behaves like P if b holds and like Q otherwise. It is read P if b else Q.

- Predicate b is on the appropriate 'state space'. Its evaluation is not an event of the conditional process.

- The process b & P (guard) is a shorthand of
  P if *b* else STOP

Conditional Choice



Conditional Choice: Laws

- The operator $\lhd b \rhd$ is idempotent, associative and distributes through operator $\lhd c \rhd$.
- Conditional choice satisfies the propositional laws

$$P \lhd true \rhd Q = P$$

$$P \lhd b \rhd Q = Q \lhd \neg b \rhd P \ldots$$

- It distributes through both internal and external choice in each argument

$$(P \sqcap Q) \lhd b \rhd R = (P \lhd b \rhd R) \sqcap (Q \lhd b \rhd R)$$

$$P \lhd b \rhd (Q \sqcap R) = (P \lhd b \rhd Q) \sqcap (P \lhd b \rhd R)$$

$$(P \,[]\, Q) \lhd b \rhd R = (P \lhd b \rhd R) \,[]\, (Q \lhd b \rhd R)$$

$$P \lhd b \rhd (Q \,[]\, R) = (P \lhd b \rhd Q) \,[]\, (P \lhd b \rhd R)$$

Conditional Choice: Laws

- General case:

  $?x{:}A \text{ -> } (P\ (x) \lhd b \rhd Q(x) = (?x{:}A \text{ -> } P(x)) \lhd b \rhd (?x{:}A \text{ -> } Q(x))$

- Conditional choice is typically used after an input event. The general law for external choice can be expressed using conditional choice:

$$(?x{:}A \text{ -> } P(x))\ []\ (?y{:}B \text{ -> } Q(y))$$

$$=$$

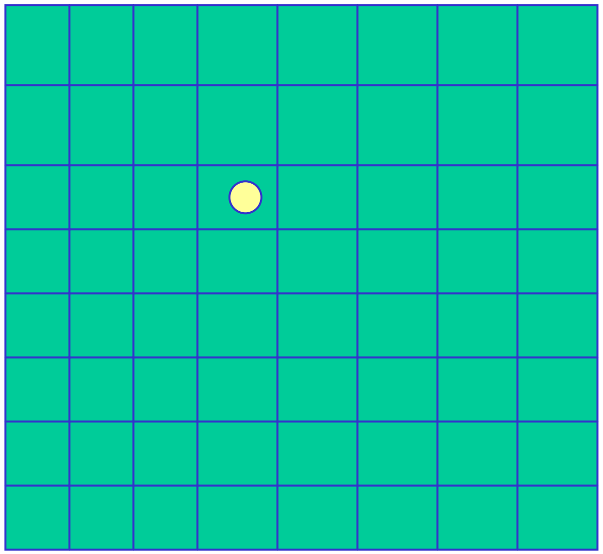
$$?z{:}\ A \cup B \text{ -> } (P(z) \sqcap Q(z))$$

$$\lhd z \in A \cap B \rhd$$

$$(P(z) \lhd z \in A \rhd Q(z))$$

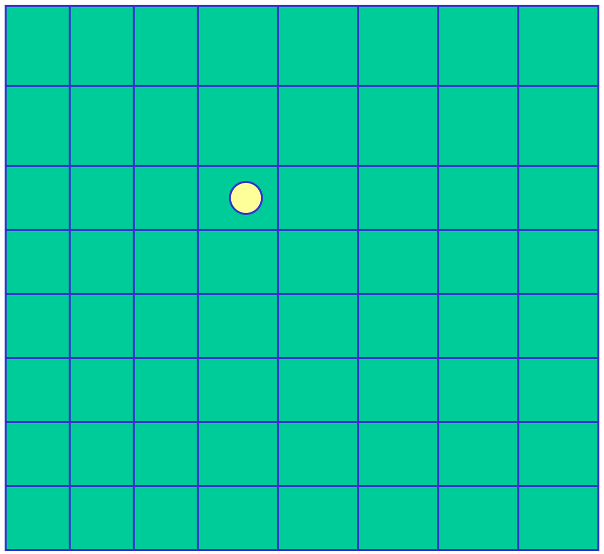
Conditional Choice: Example

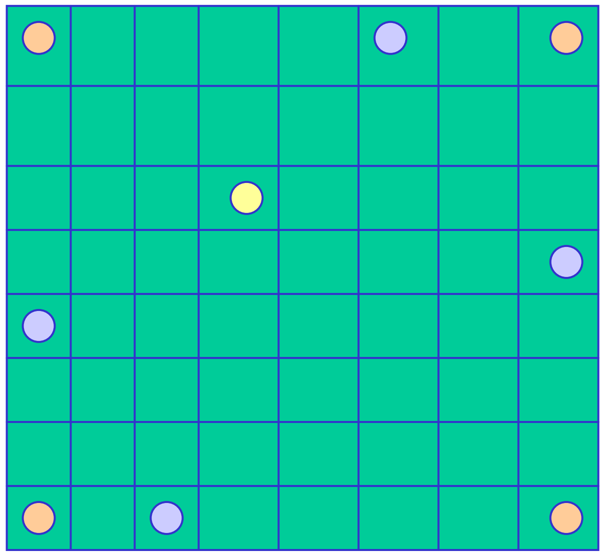
- Conditional choice can thus be used to simplify process definitions . Consider the board below. How to describe the possible movements of the yellow ball?



Conditional Choice: Example

- We need to consider nine cases.

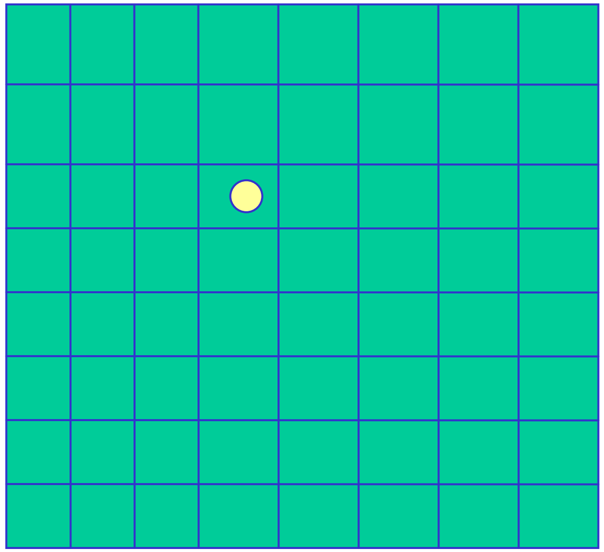




Conditional Choice: Example

- Old approach. First for an interior state $0 < x,y < 7$:

$$
\begin{aligned}
C(x,y) = (\ & up \rightarrow C(x, y+1) \\
| \ & down \rightarrow C(x, y-1) \\
| \ & left \rightarrow C(x-1, y) \\
| \ & right \rightarrow C(x+1,y)
\end{aligned}
$$



Conditional Choice: Example

- Then eight more cases !!!!

  Cases:

  $x = y = 0,$

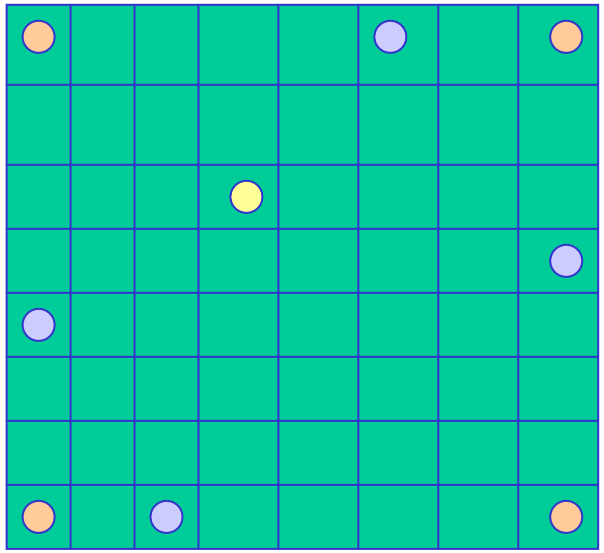
  $x = 0$ and $y = 8,$

  $x = 8$ and $y = 0,$

  $x = 8$ and $y = 8,$

  $x = 0$ and $0 < y < 7$

  …



Conditional Choice: Example

- Using conditional choice, we have a single case:

$$
\begin{aligned}
C(x,y) = (\quad & up \rightarrow C(x, y + 1) \lhd y < 7 \rhd STOP \\
[]\, & down \rightarrow C(x, y-1) \lhd 0 < y \rhd STOP \\
[]\, & left \rightarrow C(x-1, y) \lhd 0 < x \rhd STOP \\
[]\, & right \rightarrow C(x+1,y) \lhd x < 7 \rhd STOP\}
\end{aligned}
$$

Conditional Choice: Exercise

1. Write a CSP process that accepts a value and if the value is positive outputs its double and if it is negative does nothing.

Conditional Choice: Exercise

1. Write a CSP process that infinitely accepts a value. If the value is positive outputs its double, otherwise does nothing.

$$P = in\,?\,x \rightarrow ((out\,!\,2*x \text{ -> } P) \triangleleft x > 0 \triangleright P)$$